MATH 2050C Lecture 15 (Mar 16)

Limit of Functions (Ch. 4 in textbook)

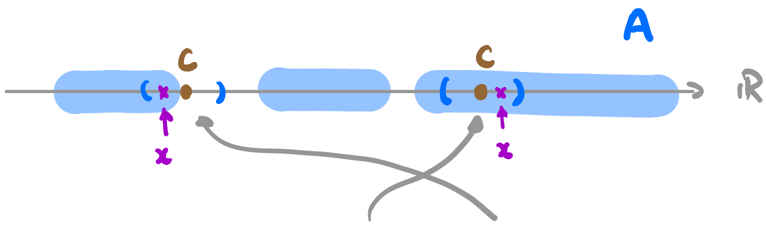
GOAL: Define $\lim_{x \to c} f(x) = L$ for functions $f : A \subseteq \mathbb{R} \to \mathbb{R}$

We shall only define $\lim_{x \to c} f(x)$ for those "c"'s which are "cluster point" of A.

so $f(x)$ is defined.

IDEA: $f(x) \overset{\varepsilon}{\approx} L$ when $x \overset{\delta}{\approx} c$ and $x \in A$

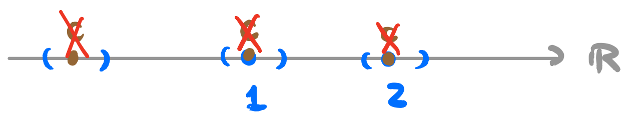
Def^{n}: Let $A \subseteq \mathbb{R}$. We say that $c \in \mathbb{R}$ is a cluster point of A iff $\forall \delta > 0$, $\exists\, x \in A$ s.t. $x \neq c$ and $|x - c| < \delta$



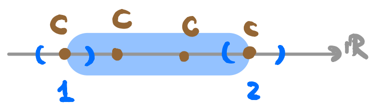
Remark: A cluster pt. $c \in \mathbb{R}$ may or may not belong to A.

Examples:

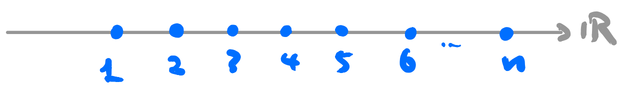
- $A = \{1, 2\}$ NO cluster pt.



- $A = (0, 1)$ Any $c \in [0, 1]$ is a cluster pt

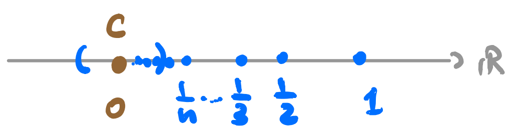


- $A = \{a_1, \dots, a_n\}$ NO cluster pt.
- $A = \mathbb{N}$ NO cluster pt.



- $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ ONLY 1 cluster pt

  $c = 0$



Prop: $c \in \mathbb{R}$ is a cluster point of A

$\iff \exists$ seq. (a_n) in A s.t. $a_n \neq c$ $\forall n \in \mathbb{N}$

and $\lim (a_n) = c$

Sketch of Proof: ($\Rightarrow$) Take $\delta_n := \frac{1}{n}$, by def^n. $\exists$ $a_n \in A$ st

$a_n \neq c$ and $|a_n - c| < \delta_n = \frac{1}{n} \xrightarrow{\text{as } n \to \infty} 0$

______ .

Common mistake in Ex. 3.3.7

$x_1 := a > 0$

$x_{n+1} := x_n + \frac{1}{x_n}$ $\forall n \in \mathbb{N}$.

Assume $\lim (x_n) =: x$ exist. $\overset{\text{not}}{\Rightarrow}$ $x = x + \frac{1}{x}$ $\Rightarrow$ $0 = \frac{1}{x}$ $\times$.

Correct unless $x \neq 0$.

We now state the most important definition for this chapter.

Def^n: Let $f : A \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a function.

Suppose $c \in \mathbb{R}$ is a cluster point of A.

We say that " f converges to $L \in \mathbb{R}$ at c ", written

" $\lim_{x \to c} f(x) = L$ " or " $f(x) \to L$ as $x \to c$ "

iff $\forall \varepsilon > 0$, $\exists$ $\delta = \delta(\varepsilon) > 0$ s.t.

$|f(x) - L| < \varepsilon$, $\forall x \in A$ where $\underbrace{0 < |x - c|}_{\text{so } x \neq c} < \delta$

Example 1: Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) := x$ for all $x \in \mathbb{R}$.

$$\lim_{x \to c} f(x) = c \qquad \forall\, c \in \mathbb{R}.$$

Pf: Any $c \in \mathbb{R}$ is a cluster pt. of $A = \mathbb{R}$.

Let $\varepsilon > 0$ be fixed but arbitrary.

Choose $\delta > 0$ st $\delta = \varepsilon$

THEN, $\forall\, x \in \mathbb{R}$, and $0 < |x - c| < \delta$, we have

$$|f(x) - c| = |x - c| < \delta = \varepsilon$$

$\blacksquare$

Remark: $\lim_{x \to c} f(x)$ may exists with f being defined at c.

E.g.) $f : A = (0,1) \to \mathbb{R}$; $f(x) := x$

$$\Rightarrow \quad \lim_{x \to 1} f(x) = 1 \notin A$$

Example 2: $\boxed{\lim_{x \to c} x^2 = c^2}$ i.e. $f : A = \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

Pf: Fix $c \in \mathbb{R}$.

Let $\varepsilon > 0$ be fixed but arbitrary.

Note: Suppose $|x - c| < 1$, then

$$|x| \le |x - c| + |c| < 1 + |c|$$

Choose $\delta := \min\left\{1, \dfrac{\varepsilon}{2(1 + 2|c|)}\right\}$

if $0 < |x - c| < \delta$, then

$$|x^2 - c^2| = |x + c| \cdot |x - c|$$
$$\le (|x| + |c|) \cdot |x - c|$$
$$\le (2|c| + \delta) \cdot \delta < \varepsilon.$$

$|x - c| < \delta \Rightarrow |x| < |c| + \delta$

THEN, $\forall x \in A = \mathbb{R}$ with $0 < |x - c| < \delta$, we have

$$|x^2 - c^2| = |x + c| \cdot |x - c| \le (1 + 2|c|)\,\delta < \varepsilon$$

Example 3: $\boxed{\lim_{x \to c} \frac{1}{x} = \frac{1}{c}}$ where $c \neq 0$.

Considering $f : A = \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) := \frac{1}{x}$

Note: Any $c \in \mathbb{R}$ is a cluster pt of A

Pf: Let's assume $c > 0$.

Let $\varepsilon > 0$ be fixed but arbitrary.

Note: If $|x - c| < \frac{c}{2}$, then

$$|x| > \frac{c}{2} > 0$$

Take $\delta := \min\left\{ \frac{c}{2}, \frac{\varepsilon c^2}{2} \right\} > 0$.

Then, $\forall x \in A$ and $0 < |x - c| < \delta$

we have

$$\left| \frac{1}{x} - \frac{1}{c} \right| = \frac{1}{|x|} \cdot \frac{1}{|c|} \cdot |x - c| < \frac{2}{c^2} \cdot \delta \le \varepsilon$$

If $0 < |x - c| < \delta$, then

$$\left| \frac{1}{x} - \frac{1}{c} \right| = \frac{|x - c|}{|xc|}$$

$$= \frac{1}{|x|} \frac{1}{|c|} |x - c| < \frac{2}{c^2} \cdot \delta \le \varepsilon$$

Need $|x|$ bdd away from 0.